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A Perturbative Approach to Fuzzifying Field Theories

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ABSTRACT

We propose a procedure for computing noncommutative corrections to the metric tensor, and apply it to scalar field theory written on coordinate patches of smooth manifolds. The procedure involves finding maps to the noncommutative plane where differentiation and integration are easily defined, and introducing a star product. There are star product independent, as well as dependent, corrections. Applying the procedure for two different star products, we find the lowest order fuzzy corrections to scalar field theory on a sphere which is sterographically projected to the plane.

1 Introduction

Currently, field theories have been successfully written down on only a handful of noncommuting manifolds. Besides being of intrinsic interest, the search for new noncommutative geometries is relevant for quantum gravity and also string theory, where one expects to have to sum over all such geometries. Moreover, given a field theory written on an arbitrary curved manifold there is no canonical procedure for making the theory noncommutative. The inverse problem can also be problematic. Here one should distinguish between classical and quantum noncommutative field theories, because the commutative limit and the classical limit may not commute. Starting from noncommutative quantum theory it has been noted that a phase transition can occur in passing to the commutative theory.[1] For such theories the commutative limit is singular. On the other hand, it should always be possible to take the commutative limit of classical noncommutative field theory by simply setting the noncommutative parameter to zero.

In this article we shall be concerned with classical noncommutative field theory, with the aim of developing a systematic procedure for computing noncommutative or fuzzy corrections to classical field theories written on coordinate patches of arbitrary smooth manifolds. We report on progress in this direction for the case of scalar field theory in two dimensions. The approach taken does not involve the more formal aspects of noncommutative geometry. We start with some noncommutative associative algebra $\mathcal{A}$. There are a number of obstacles to constructing a noncommutative space from $\mathcal{A}$. In addition to insuring that the Jacobi identity is satisfied, there is the problem of defining the notion of derivations and integration on this space. As we shall only be interested in two-dimensional theories the Jacobi identity is trivially satisfied. Two-dimensional examples where derivations and integration can be defined are the noncommutative plane, noncommutative torus, fuzzy sphere[2] and fuzzy disc[3]. In these examples derivatives are inner, i.e. they are obtained with the adjoint action of the generators, and integration is defined with the usual trace. Of interest here will be the noncommutative plane which is generated by creation and annihilation operators, $\mathbf{a}^\dagger$ and $\mathbf{a}$, respectively

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1\!\!1 \;, \tag{1.1}$$

$1\!\!1$ being the unit operator, with derivatives of any function Φ of $\mathbf{a}^\dagger$ and $\mathbf{a}$ given by

$$\bar{\nabla}\Phi = [\mathbf{a}\;, \Phi] \qquad \nabla\Phi = [\Phi, \mathbf{a}^\dagger] \tag{1.2}$$

They satisfy Leibinz rule along with $[\nabla, \bar{\nabla}] = 0$, $\nabla\mathbf{a} = \bar{\nabla}\mathbf{a}^\dagger = 1\!\!1$ and $\bar{\nabla}\mathbf{a} = \nabla\mathbf{a}^\dagger = 0$. The action of a free massless scalar field Φ on the noncommutative plane can be written as

$$S_0 \;\; = \;\; \pi \;\mathrm{Tr}\; \nabla\Phi \; \bar{\nabla}\Phi \tag{1.3}$$

More generally, if we are given a pair of generators, $\mathbf{z}$ and its hermitean conjugate $\mathbf{z}^\dagger$, satisfying arbitrary commutation relations

$$[\mathbf{z}, \mathbf{z}^\dagger] = \Theta(\mathbf{z}, \mathbf{z}^\dagger) \;\;, \tag{1.4}$$

it is in general unclear how to define derivations or integration.* If there exists an algebraic map from $\mathbf{z}$ and $\mathbf{z}^\dagger$ to generators $\mathbf{a}$ and $\mathbf{a}^\dagger$ of the noncommutative plane, we can define derivatives of functions as in (1.2), and then apply the inverse map back to $\mathbf{z}$ and $\mathbf{z}^\dagger$. This was proposed previously [5], but discussion was restricted to nonsingular maps. It is the noncommutative analog of mapping from generalized coordinates on a two dimensional manifold $\mathcal{M}$ to the plane. In the commutative setting the existence of a globally defined map to the plane implies a flat geometry, as opposed to manifolds $\mathcal{M}$ with nonvanishing curvature where only local maps to the plane are defined. Since here we would like to allow for the possibility of recovering the latter in the commutative limit, we should admit 'local' maps in the noncommutative setting. Local in noncommutative theories may be defined by a restriction of the spectrum of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$. We note that singular maps need not imply a singular field theory. From (1.2), $\mathbf{a}$ and $\mathbf{a}^\dagger$ appear in the derivatives of the fields using the commutator, and so only commutators of $\mathbf{a}$ and $\mathbf{a}^\dagger$ with the fields need be well defined. This may be possible to arrange even if $\mathbf{a}$ and $\mathbf{a}^\dagger$ are not well defined functions of $\mathbf{z}$ and $\mathbf{z}^\dagger$ by imposing suitable boundary conditions on the fields. This was done previously in [6].

As is well known, the operator algebra can be realized on the (commutative) plane with the use of a star product. For this, operators are mapped to functions ('symbols') on the plane, and the product of two operators is mapped to the star product of symbols. Actions such as (1.3) can then be re-expressed as integrals on the plane. Here instead of writing in terms of symbols of the generators $\mathbf{a}$ and $\mathbf{a}^\dagger$ of the noncommutative plane, we write the result in terms of symbols of the operators $\mathbf{z}$ and $\mathbf{z}^\dagger$ appearing in (1.4). In the commutative limit, the latter symbols reduce to coordinates, denote them by x^μ, that can be used to parametrize a coordinate patch of some manifold $\mathcal{M}$. By performing a derivative expansion, which is equivalent to an expansion in the noncommutativity parameter, we can then obtain noncommutative corrections to the commutative action. In the case of the free scalar field $\phi(x)$ the commutative action on any coordinate patch $\mathcal{P}$ is

$$\mathcal{S}_0^0 = \frac{1}{2}\int_{\mathcal{P}} d^2x\sqrt{g}\, g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu}\ , \tag{1.5}$$

g and $g^{\mu\nu}$ being the determinant and inverse components, respectively, of the metric $[g_{\mu\nu}]$. The procedure for obtaining noncommutative corrections depends on the choice of star product. Star products are equivalent if they are related by a Kontsevich map[7], and so the dynamics computed to all orders should be identical for equivalent star products. Common choices for the star product on the noncommutative plane are the Moyal-Weyl and the Voros. The Voros star product is based on coherent states which diagonalize $\mathbf{a}$. An alternative star product was developed in [8]. It uses an overcomplete basis of states developed in [9] which instead diagonalize $\mathbf{z}$. The Voros star product and the one in [8] have certain advantages and disadvantages. We shall use both of the star products.

As an example we consider the fuzzy sphere. In [8] we wrote down a noncommutative analogue of the stereographic projection of a sphere in terms of operators $\mathbf{z}$ and $\mathbf{z}^\dagger$ satisfying

*Derivations can be obtained in special cases.[4]

(1.4) for some $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$. The noncommutative stereographic projection is nonsingular, although the map to the noncommutative plane is not. For the latter it is necessary to define a truncated harmonic oscillator Hilbert space. Consequently, we must modify the definition of coherent states and the star products in this case.

We introduce the general formalism in section 2, and discuss the stereographically projected fuzzy sphere in section 3. In section 4 we conclude by mentioning possible generalizations of this work.

2 General Framework

We first review free scalar field theory written on a local coordinate patch in two dimensions. We include a Poisson structure and a map to the plane. We then consider the noncommutative version of the theory and apply the above procedure to compute lowest order corrections.

2.1 Commutative Theory

Say $\mathcal{P}$ is a coordinate patch for some two dimensional manifold $\mathcal{M}$. On $\mathcal{P}$ denote the metric by $g_{\mu\nu}(x)$, x^μ, $\mu = 1, 2$, being the coordinates. Upon introducing zweibein fields $e^a{}_\mu(x)$, $a = 1, 2$ being the flat index, the metric can be expressed by

$$g_{\mu\nu}(x) = e^a{}_\mu(x)\, e^a{}_\nu(x) \tag{2.1}$$

The zweibein fields transform vectors to a local orthogonal frame $\mathcal{O}$, the latter associated with the metric δ_{ab}. If $\{\frac{\partial}{\partial x^\mu}\}$ and $\{\frac{\partial}{\partial y^a}\}$ are the set of basis vectors of $\mathcal{P}$ and $\mathcal{O}$, respectively, then

$$\frac{\partial}{\partial x^\mu} = e^a{}_\mu(x)\, \frac{\partial}{\partial y^a} \tag{2.2}$$

Alternatively, the components v^μ and u^a of tangent vector V written in the two bases, respectively, are related by $u^a = e^a{}_\mu(x)\, v^\mu$. The set of all orthogonal frames $\{\mathcal{O}\}$ are related by local orthogonal transformations. So if u'^a are components of V in orthogonal frame $\mathcal{O}'$ then

$$u'^a = \lambda^a{}_b(x) u^b\ , \tag{2.3}$$

$\lambda(x) = [\lambda^a{}_b(x)]$ being an orthogonal matrix. In the tetrad formalism one normally also introduces the spin connection. However here since we shall only be interested in scalar field theory we do not require this additional structure.

Next we assume a nonsingular Poisson structure on $\mathcal{P}$. So we define a Poisson tensor with components $\theta^{\mu\nu}(x) = -\theta^{\nu\mu}(x)$. The Poisson bracket of any two functions f and g on $\mathcal{P}$ is

$$\{f, g\}(x) = \theta^{\mu\nu}(x)\, \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} \tag{2.4}$$

So

$$\{x^\mu, x^\nu\}(x) = \theta^{\mu\nu}(x) \tag{2.5}$$

This is analogous to the commutation relations (1.4). Now choose

$$\theta^{\mu\nu}(x) = \theta(x)\epsilon^{\mu\nu}\ , \qquad \theta(x) = \frac{1}{\sqrt{g(x)}} \tag{2.6}$$

Then in local orthogonal frame $\mathcal{O}$ the Poisson tensor is mapped to the constant antisymmetric tensor

$$\tilde{\theta}^{ab} = \theta^{\mu\nu}(x)e^a{}_\mu(x)e^b{}_\nu(x) = \epsilon^{ab} \tag{2.7}$$

This is true for the set of all locally orthogonal frames $\{\mathcal{O}\}$ (in two dimensions) since the constant tensor is preserved under local orthogonal transformations (2.3). (This is not the case in higher dimensions.) Finally, in order that $\mathcal{O}$ is a noncommutative plane at lowest order, we need to impose that it defines a coordinate bases. Thus the map from $\mathcal{P}$ to a $\mathcal{O}$ should be a coordinate map. This is since in the noncommutative theory we need an explicit map between the noncommutative analogues of the coordinates x^μ on $\mathcal{P}$ and the coordinates y^a in frame $\mathcal{O}$. The map is in general only valid on some open region σ on $\mathcal{O}$. So $e^a{}_\mu(x) = \frac{\partial y^a}{\partial x^\mu}$ and as a result

$$\{y^a, y^b\} = \epsilon^{ab}\ , \tag{2.8}$$

which is the analogue of (1.1).

The basis vectors of $\mathcal{O}$ can be expressed in terms of the Poisson bracket

$$\frac{\partial}{\partial y^a} = [\tilde{\theta}^{-1}]_{ab}\ \{y^b\ ,\ \ \}\ , \tag{2.9}$$

in analogy to the inner derivatives (1.2), and then so can the basis vectors of $\mathcal{P}$

$$\frac{\partial}{\partial x^\mu} = -e^a{}_\mu(x)\ \epsilon_{ab}\ \{y^b\ ,\ \ \} \tag{2.10}$$

We can apply this to the case of a scalar field theory on $\mathcal{M}$. If ϕ is a real function of coordinates y of $\mathcal{O}$, then the standard free action on the region σ on $\mathcal{O}$ is given by

$$\mathcal{S}_0^0 = \frac{1}{2}\int_\sigma d^2y\ \{y^a\ ,\phi\}\{y^a\ ,\phi\} \tag{2.11}$$

Upon doing a change of variables one gets (1.5) on the corresponding region σ' of $\mathcal{P}$. The action (1.5) is exact for fields with nonvanishing support only on σ'.

2.2 Noncommutative analogue

The prescription to go to the noncommutative plane is to replace real functions by hermitean operators, Poisson brackets with $-i\theta_0$ times the commutator, and the integration $\int d^2y$ by

$2\pi\theta_0$ Tr. θ_0 is the noncommutativity parameter. So if we replace y^a by hermitean operators Y^a, they satisfy the Heisenberg algebra

$$[Y^a, Y^b] = i\epsilon^{ab}\theta_0 \mathbb{1} \,, \tag{2.12}$$

which defines the noncommutative plane. Alternatively, we can define creation and annihilation operators $\mathbf{a}^\dagger$ and $\mathbf{a}$, respectively

$$\mathbf{a} = \frac{Y^1 + iY^2}{\sqrt{2\theta_0}} \,, \qquad \mathbf{a}^\dagger = \frac{Y^1 - iY^2}{\sqrt{2\theta_0}} \,, \tag{2.13}$$

satisfying (1.1). Next introduce the analogue Φ of the scalar field ϕ. It is a hermitean function on the noncommutative plane defined in the enveloping algebra generated by Y^a. The free field action (2.11) goes to

$$S_0 \quad = \quad -\frac{\pi}{\theta_0}\,\mathrm{Tr}_\Sigma\,[Y^a\,,\Phi]\,[Y^a\,\Phi]\,, \tag{2.14}$$

or equivalently, (1.3). The trace in (2.14) over all fields Φ would imply that the dynamics is on the noncommutative plane. However, we would like to allow for dynamics on other two dimensional noncommutative manifolds M. For this reason we inserted the subscript Σ on the trace, which indicates that the expression (2.14) is valid for a restricted set of fields Φ, and it is analogous to the restriction in (2.11). The latter is for fields ϕ to have nonvanishing support only on some small region σ, which we can assume is centered around the origin of coordinate system $\{y^a\}$. Only in the case of a flat geometry can we take σ to be all of R^2. Just as (2.11) is not 'globally' valid for an arbitrary (commutative) manifold $\mathcal{M}$, the expression (2.14) cannot be 'globally' valid for an arbitrary noncommutative manifold M. We should then restrict in (2.14) to fields which are defined in a small region Σ of the noncommutative plane. By this we mean Φ acts nontrivially only on eigenstates $|n>$ of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$ with eigenvalues n sufficiently small $0 \leq n < n_0$. Say S is the exact expression for the scalar field on M. Then (2.14) should be a reasonable approximation of S for the case where Φ vanishes outside Σ. This will be demonstrated for the fuzzy sphere in the next section. For fuzzy spaces the Hilbert space is, by definition, finite dimensional. Moreover, for large dimension N of the Hilbert space, θ_0 is inversely related to N, so that the commutative limit $\theta_0 \to 0$ corresponds to $N \to \infty$. In that case we can define small as $n_0 << N$. The approximation is improved by including higher order corrections from S. We assume that the next order corrections S_1 to (2.14) go like θ_0, or equivalently $1/N$. This is the case for the fuzzy sphere. More generally, we expand the exact expression S for the action according to

$$S = S_0 + S_1 + S_2 + \ldots \,, \tag{2.15}$$

where S_{m+1}/S_m goes like θ_0.

Instead of starting with (2.12), we can examine the more general algebra generated by $\mathbf{z}$ and its hermitean conjugate $\mathbf{z}^\dagger$, with commutator given in (1.4) for some arbitrary function $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$. Say that the lowest order of the parameter θ_0 in $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$ is linear, and in the commutative limit

$\theta_0 \to 0$, the symbol of $\Theta(\mathbf{z}, \mathbf{z}^\dagger)$ tends to $\theta_0\theta(x)$, where $\theta(x)$ was given in (2.6). So (1.4) is the noncommutative analogue of (2.5), with $\mathbf{z}$ and $\mathbf{z}^\dagger$ the noncommutative analogues of $x_1 + ix_2$ and $x_1 - ix_2$, respectively, and we recover the coordinate patch $\mathcal{P}$ in the commutative limit. We already assumed the existence of a coordinate map from $\mathcal{P}$ to an orthogonal frame $\mathcal{O}$, and so now we assume that a noncommutative analogue of this is true; i.e. that there is an algebraic map from $\mathbf{z}$ and $\mathbf{z}^\dagger$ to Y^a (or, equivalently, $\mathbf{a}$ and $\mathbf{a}^\dagger$). The map is in general only 'local', meaning that it is only defined on eigenstates $|n>$ of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$ with eigenvalues n sufficiently small $0 \leq n < n_0$. The map allows us to re-express the action S 'locally' in terms of $\mathbf{z}$ and $\mathbf{z}^\dagger$. Next one can introduce a star product, replacing product of two operators by the star product of their corresponding symbols, and the trace by integration with respect to some measure. Then the expansion (2.15) maps to a corresponding expansion of integrals on the plane

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + ... \tag{2.16}$$

We shall express $\mathcal{S}_m$ as integrals in the covariant symbols ζ and $\bar{\zeta}$ of $\mathbf{z}$ and $\mathbf{z}^\dagger$, respectively. There is then a further θ_0 expansion that one can do since the star product contains θ_0 to all orders. So

$$\mathcal{S}_m = \mathcal{S}_m^0 + \mathcal{S}_m^1 + \mathcal{S}_m^2 + ... \ , \tag{2.17}$$

where S_m^{p+1}/S_m^p goes like θ_0. The zeroth order term $\mathcal{S}_m^0$ in the expansion is of order θ_0^m. It is associated with the ordinary product, and so it is star product independent. In particular, $\mathcal{S}_0^0$ is the commutative result, i.e. (1.5). More generally, $\mathcal{S}_m^p$ is of order θ_0^{m+p}. As we shall only be interested in the leading θ_0 corrections we will only compute $\mathcal{S}_1^0$ and $\mathcal{S}_0^1$, the former being star-product independent. To compute it we will need the exact expression for the action (or Laplacian). For the latter we must specify the star-product. Below we shall compute $\mathcal{S}_0^1$ for the case of two different star products: 1) The Voros star product and 2) the generalized star product of ref. [8].

2.3 Voros star product

The Voros product is based on standard coherent states on the complex plane. Denote them by $|\alpha>_V$, α being the coordinate on the complex plane, satisfying ${}_V\!<\alpha|\alpha>_V = 1$ and

$$\mathbf{a}|\alpha>_V = \alpha|\alpha>_V \tag{2.18}$$

The covariant symbol $\mathcal{A}_V(\alpha, \bar{\alpha})$ of an operator A is a function on the complex plane given by the matrix element $\mathcal{A}_V(\alpha, \bar{\alpha}) = {}_V\!<\alpha|A|\alpha>_V$. The star product $\star_V$ between any two covariant symbols $\mathcal{A}_V(\alpha, \bar{\alpha})$ and $\mathcal{B}_V(\alpha, \bar{\alpha})$ associated with operators A and B is defined to be the covariant symbol of the product of operators:

$$[\mathcal{A}_V \star_V \mathcal{B}_V](\alpha, \bar{\alpha}) \ = {}_V\!<\alpha|AB|\alpha>_V$$

and here

$$\star_V = \exp \ \frac{\overleftarrow{\partial}}{\partial\alpha} \ \frac{\overrightarrow{\partial}}{\partial\bar{\alpha}}$$

Derivatives in α and $\bar\alpha$ are given by

$$\begin{aligned}\frac{\partial}{\partial\alpha}\ \mathcal{A}_V(\alpha,\bar\alpha) &= {}_V\!<\alpha|[A,\mathbf{a}^\dagger]|\alpha>_V\\ \frac{\partial}{\partial\bar\alpha}\ \mathcal{A}_V(\alpha,\bar\alpha) &= {}_V\!<\alpha|[\mathbf{a},A]|\alpha>_V\end{aligned}\tag{2.19}$$

Defining ϕ_V to be the covariant symbol of the field operator Φ, the action (1.3) can be mapped to

$$\mathcal{S}_0=\pi\int d\mu_V(\alpha,\bar\alpha)\left[\frac{\partial\phi_V}{\partial\alpha}\ \star_V\ \frac{\partial\phi_V}{\partial\bar\alpha}\right](\alpha,\bar\alpha)\ ,\tag{2.20}$$

where $d\mu_V(\alpha,\bar\alpha)$ is the measure satisfying the partition of unity $\int d\mu_V(\alpha,\bar\alpha)\ |\alpha>_V\ {}_V\!<\alpha|=1\!\!1$. For the standard coherent states it is

$$d\mu_V(\alpha,\bar\alpha)\ =\frac{i}{2\pi}d\alpha\wedge d\bar\alpha\tag{2.21}$$

In the commutative limit $\theta_0\to0$, or $\alpha,\bar\alpha\to\infty$, and the Lagrangian in (2.20) reduces to that of the free scalar field (2.11) written in an orthogonal frame $\mathcal{O}$. For this use $y^1=\sqrt{\frac{\theta_0}{2}}\ (\bar\alpha+\alpha)$ and $y^2=i\sqrt{\frac{\theta_0}{2}}\ (\bar\alpha-\alpha)$, which are the covariant symbols of the generators Y^1 and Y^2 of the noncommutative plane.

Finally we wish to map back to the coordinate patch $\mathcal{P}$. We take it to be spanned by the covariant symbols ζ and $\bar\zeta$ of $\mathbf{z}$ and $\mathbf{z}^\dagger$, respectively, as they have the correct commutative limit. They are local functions of the covariant symbols α and $\bar\alpha$ of $\mathbf{a}$ and $\mathbf{a}^\dagger$. These functions must then be inverted to express the system in terms of ζ and $\bar\zeta$. To compare tangent vectors in $\mathcal{P}$ and $\mathcal{O}$ one can define the analogue of an inverse zweibein matrix

$$h_V=\begin{pmatrix}h^\zeta{}_\alpha & h^\zeta{}_{\bar\alpha}\\ h^{\bar\zeta}{}_\alpha & h^{\bar\zeta}{}_{\bar\alpha}\end{pmatrix}=\begin{pmatrix}\frac{\partial\zeta}{\partial\alpha} & \frac{\partial\zeta}{\partial\bar\alpha}\\ \frac{\partial\bar\zeta}{\partial\alpha} & \frac{\partial\bar\zeta}{\partial\bar\alpha}\end{pmatrix}\ ,\tag{2.22}$$

which goes like $\sqrt{\theta_0}$ in the commutative limit. So on $\mathcal{P}$ the free scalar field is $\phi(\zeta,\bar\zeta)=\phi_V(\alpha(\zeta,\bar\zeta),\bar\alpha(\zeta,\bar\zeta))$, and the action (2.20) becomes

$$\mathcal{S}_0=\frac{i}{2}\int\frac{d\zeta\wedge d\bar\zeta}{\det h_V}\ \mathcal{L}_0\ ,$$

$$\mathcal{L}_0\ =\ \sum_{n=0}^{\infty}\frac{1}{n!}\left[\left(h^\zeta{}_\alpha\frac{\partial}{\partial\zeta}+h^{\bar\zeta}{}_\alpha\frac{\partial}{\partial\bar\zeta}\right)^{n+1}\phi\right]\left[\left(h^\zeta{}_{\bar\alpha}\frac{\partial}{\partial\zeta}+h^{\bar\zeta}{}_{\bar\alpha}\frac{\partial}{\partial\bar\zeta}\right)^{n+1}\phi\right]\tag{2.23}$$

The measure, as well as the terms in the sum, in general contain different powers of the noncommutativity parameter θ_0. To write the result as an expansion in θ_0 one has to expand the components of h_V. At lowest order in θ_0 we recover the commutative result (1.5), i.e.

$$\mathcal{S}_0^0=\frac{i}{2}\int d\zeta\wedge d\bar\zeta\ \sqrt{g}\ \mathcal{L}_0^0\ ,$$

$$\mathcal{L}_0^0=g^{\zeta\zeta}\ (\partial\phi)^2+g^{\bar\zeta\bar\zeta}\ (\bar\partial\phi)^2+2g^{\zeta\bar\zeta}\ |\partial\phi|^2\tag{2.24}$$

Here we set $\partial=\partial/\partial\zeta$ and $\bar\partial=\partial/\partial\bar\zeta$ and

$$\begin{pmatrix}g^{\zeta\zeta} & g^{\zeta\bar\zeta}\\ g^{\bar\zeta\zeta} & g^{\bar\zeta\bar\zeta}\end{pmatrix}=\begin{pmatrix}h^\zeta{}_\alpha h^\zeta{}_{\bar\alpha} & \frac12(h^\zeta{}_\alpha h^{\bar\zeta}{}_{\bar\alpha}+h^{\bar\zeta}{}_\alpha h^\zeta{}_{\bar\alpha})\\ \frac12(h^\zeta{}_\alpha h^{\bar\zeta}{}_{\bar\alpha}+h^{\bar\zeta}{}_\alpha h^\zeta{}_{\bar\alpha}) & h^{\bar\zeta}{}_\alpha h^{\bar\zeta}{}_{\bar\alpha}\end{pmatrix}\tag{2.25}$$

2.4 Generalized star product

The advantage of the Voros star product is that it has a simple closed form expression. On the other hand, the above procedure was complicated by the fact that we had to invert the functions $\zeta(\alpha,\bar\alpha)$ and $\bar\zeta(\alpha,\bar\alpha)$ in order to do the change of variables in (2.20). This complication is avoided if one can start with a star product based on an overcomplete set of states $\{|\zeta>\}$ which diagonalize $\mathbf{z}$ rather than $\mathbf{a}^\dagger$

$$\mathbf{z}|\zeta>=\zeta|\zeta> \,, \tag{2.26}$$

ζ denoting a complex variable. The states $\{|\zeta>\}$ were found in [9] and are a nonlinear deformations of standard coherent states $\{\widetilde{|\alpha>}\}$ on the complex plane. The covariant symbol of operator A, B,... are given by $\mathcal{A}(\zeta,\bar\zeta) =<\zeta|A|\zeta>$, $\mathcal{B}(\zeta,\bar\zeta) =<\zeta|B|\zeta>$,... , and their star product by $[\mathcal{A}\star\mathcal{B}](\zeta,\bar\zeta)) =<\zeta|AB|\zeta>$. From $<\zeta|\zeta>=1$ it follows that the complex coordinates ζ and its complex conjugate $\bar\zeta$ are the symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$. The action can then be directly written in terms of functions of these covariant symbols.

The disadvantage of this approach is that the expression for the star product is not simple unlike the case of the Voros star product. The expression was obtained in [8]. Using the property that the ratio $<\zeta|A|\eta>/<\zeta|\eta>$ is analytic in η and anti-analytic in ζ, one gets

$$[\mathcal{A}\star\mathcal{B}](\zeta,\bar\zeta)=\mathcal{A}(\zeta,\bar\zeta)\quad\int d\mu(\eta,\bar\eta)\quad :\exp\frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta):\quad|<\zeta|\eta>|^2\quad :\exp{(\bar\eta-\bar\zeta)}\frac{\overrightarrow{\partial}}{\partial\bar\zeta}:\quad\mathcal{B}(\zeta,\bar\zeta)\,, \tag{2.27}$$

where $d\mu(\zeta,\bar\zeta)$ is the appropriate measure on the complex plane satisfying the partition of unity $\int d\mu(\zeta,\bar\zeta)\,|\zeta><\zeta| = 1\!\!1$. The colons in (2.27) denote an ordered exponential, with the derivatives ordered to the right in each term in the Taylor expansion of $\exp{(\eta-\zeta)}\frac{\overrightarrow{\partial}}{\partial\zeta}$, and to the left in each term in the Taylor expansion of $\exp\frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta)$. Thus

$$\begin{aligned} :\exp{(\bar\eta-\bar\zeta)}\frac{\overrightarrow{\partial}}{\partial\bar\zeta}: \quad &= \quad 1+(\bar\eta-\bar\zeta)\frac{\overrightarrow{\partial}}{\partial\bar\zeta}+\frac12(\bar\eta-\bar\zeta)^2\frac{\overrightarrow{\partial^2}}{\partial\bar\zeta^2}+\cdots \\ :\exp\frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta): \quad &= \quad 1+\frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta)+\frac12\frac{\overleftarrow{\partial^2}}{\partial\zeta^2}(\eta-\zeta)^2+\cdots \end{aligned} \tag{2.28}$$

The commutative limit is obtained by performing a derivative expansion, which was done in [6]. One obtains the following leading three terms acting on functions of ζ and $\bar\zeta$:

$$\star \;=\; 1 \;+\; \frac{\overleftarrow{\partial}}{\partial\zeta}\,\theta_S(\zeta,\bar\zeta)\,\frac{\overrightarrow{\partial}}{\partial\bar\zeta} \;+\; \frac14\left[\frac{\overleftarrow{\partial^2}}{\partial\zeta^2}\;\frac{\overrightarrow{\partial}}{\partial\bar\zeta}\theta_S(\zeta,\bar\zeta)^2\frac{\overrightarrow{\partial}}{\partial\bar\zeta}+\frac{\overleftarrow{\partial}}{\partial\zeta}\theta_S(\zeta,\bar\zeta)^2\;\frac{\overleftarrow{\partial}}{\partial\zeta}\;\frac{\overrightarrow{\partial^2}}{\partial\bar\zeta^2}\right] \;+\;\cdots \tag{2.29}$$

where $\theta_S(\zeta,\bar\zeta)$ is the symbol of $\Theta(\mathbf{z},\mathbf{z}^\dagger)$. [The S subscript distinguishes it from the classical value.] At lowest order, $\theta_S(\zeta,\bar\zeta)\rightarrow\theta_0\;\theta(\zeta,\bar\zeta)$. Since the limit is linear in θ_0, the derivative expansion is also an expansion in θ_0. The Poisson bracket is recovered from the star commutator

[†]For this it is necessary that the algebra (1.4) have only infinite dimensional representations.

at leading order. [The derivation of (2.29) requires (2.26). The latter is not true for the case of the fuzzy sphere, as we shall see in the next section. Thus (2.29) gets modified for that case.]

Now we return to the free scalar field with action (2.14). We define symbol of the field Φ as $\phi(\zeta,\bar{\zeta}) =< \zeta|\Phi|\zeta >$. In oder to compute its derivatives we define the symbols $y^a(\zeta,\bar{\zeta}) = < \zeta|Y^a|\zeta >$ of Y^a, $< \zeta|\mathbf{a}|\zeta >$ of $\mathbf{a}$ and $< \zeta|\mathbf{a}^\dagger|\zeta >$ of $\mathbf{a}^\dagger$. [The latter two are, in general, not the same as $\alpha(\zeta,\bar{\zeta})$ and $\bar{\alpha}(\zeta,\bar{\zeta})$ computed previously by taking the inverse of the covariant symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$ with respect to the standard coherent states $|\alpha >_V$.] Now the action can be expressed as

$$\mathcal{S}'_0 = \pi\theta_0 \int_\sigma d\mu(\zeta,\bar{\zeta})\; \mathcal{L}'_0 \;, \qquad \mathcal{L}'_0 = -\frac{1}{\theta_0^2}(y^a \star \phi - \phi \star y^a)^{\star 2} \;, \tag{2.30}$$

where $\mathcal{A}^{\star 2} = \mathcal{A}\star\mathcal{A}$ and we use the prime to distinguish this result from the one in the previous subsection. (2.30) reduces to (1.5) in the commutative limit $\theta_0 \to 0$; i.e., if we expand $\mathcal{L}'_0$ in θ_0 the zeroth order term $\mathcal{L}^0_0$ is again given by (2.24). Then in comparing with (1.5),

$$2\pi\theta_0\; d\mu(\zeta,\bar{\zeta}) \to i\sqrt{g}\; d\zeta\wedge d\bar{\zeta} \;, \qquad \text{as } \; \theta_0 \to 0 \tag{2.31}$$

Going beyond the lowest order, we have to expand the Lagrangian density

$$\mathcal{L}'_0 = \mathcal{L}^0_0 + \mathcal{L}^{1'}_0 + \mathcal{L}^{2'}_0 + \ldots \;,$$

as well as the measure. Like in the previous subsection, the first order correction contains quadratic, cubic and quartic terms in derivatives of ζ and $\bar{\zeta}$ of $\phi(\zeta,\bar{\zeta})$, but the coefficients of these terms can differ from the previous results. The coefficients can be expressed in terms of θ_S and

$$h_a = \frac{\theta_S}{\theta_0}\partial y^a \qquad \bar{h}_a = \frac{\theta_S}{\theta_0}\bar{\partial} y^a \;, \tag{2.32}$$

and their derivatives. At lowest order, h_a and $\bar{h}_a$ are inverse zweibein components, and $\theta_S = -i\theta_0\epsilon_{ab}h_a\bar{h}_b$. The quadratic terms are

$$G^{\zeta\zeta}\;(\partial\phi)^2 + G^{\bar{\zeta}\bar{\zeta}}\;(\bar{\partial}\phi)^2 + 2G^{\zeta\bar{\zeta}}\;|\partial\phi|^2 \;, \tag{2.33}$$

where up to first order in θ_S

$$\begin{aligned} G^{\bar{\zeta}\bar{\zeta}} &= \overline{G^{\zeta\zeta}} = -h_a \star h_a - \theta_S\bar{\partial}\theta_S h_a \partial(\theta^{-1}h_a) \\ 2G^{\zeta\bar{\zeta}} &= h_a\star\bar{h}_a + \bar{h}_a\star h_a + \theta_S\partial\theta_S h_a\bar{\partial}(\theta_S^{-1}\bar{h}_a) + \theta_S\bar{\partial}\theta_S\bar{h}_a\partial(\theta_S^{-1}h_a) \end{aligned} \tag{2.34}$$

The lowest order terms correspond to the components of g^{-1}. We can then interpret $G^{\zeta\zeta}$, $G^{\zeta\bar{\zeta}}$ and $G^{\bar{\zeta}\bar{\zeta}}$ as corrections to the inverse metric. The cubic terms are

$$G^{\zeta,\zeta\zeta}\;\partial\phi\partial^2\phi + G^{\bar{\zeta},\zeta\zeta}\;\bar{\partial}\phi\partial^2\phi + G^{\zeta,\zeta\bar{\zeta}}\;\partial\phi\partial\bar{\partial}\phi \;\; + \;\; \text{complex conjugate} \;, \tag{2.35}$$

where

$$G^{\zeta,\zeta\zeta} \;\; = \;\; -\theta_S\bar{\partial}(\bar{h}_a\bar{h}_a)$$

$$
\begin{aligned}
G^{\bar{\zeta},\zeta\zeta} &= \theta_S \bar{\partial}(h_a \bar{h}_a) \\
G^{\zeta,\zeta\bar{\zeta}} &= \theta_S h_a \bar{\partial} \bar{h}_a - \theta_S \bar{h}_a \partial \bar{h}_a
\end{aligned}
\tag{2.36}
$$

The quartic terms are

$$
G^{\zeta\zeta,\bar{\zeta}\bar{\zeta}}\ |\partial^2\phi|^2 + G^{\zeta\zeta,\zeta\bar{\zeta}}\ \partial^2\phi\partial\bar{\partial}\phi + G^{\bar{\zeta}\bar{\zeta},\zeta\bar{\zeta}}\ \bar{\partial}^2\phi\partial\bar{\partial}\phi + G^{\zeta\bar{\zeta},\zeta\bar{\zeta}}\ (\partial\bar{\partial}\phi)^2\ , \tag{2.37}
$$

where

$$
\begin{aligned}
G^{\zeta\zeta,\bar{\zeta}\bar{\zeta}} &= G^{\zeta\bar{\zeta},\zeta\bar{\zeta}} = \theta_S \bar{h}_a h_a \\
G^{\bar{\zeta}\bar{\zeta},\zeta\bar{\zeta}} &= \overline{G^{\zeta\zeta,\zeta\bar{\zeta}}} = -\theta_S h_a h_a
\end{aligned}
\tag{2.38}
$$

To complete the expansion in θ_0 we will need the series expansion of $\theta_S(\zeta,\bar{\zeta})$, as well as of the inverse zweibein h_a about the commutative answers. Another issue is the measure, which is defined to satisfy the partition of unity

$$
\int d\mu(\zeta,\bar{\zeta}) |\zeta><\zeta| = 1\!\!1 \tag{2.39}
$$

For the case where the map from $\mathbf{a}$ and $\mathbf{a}^\dagger$ to $\mathbf{z}$ and $\mathbf{z}^\dagger$ is of the form

$$
\mathbf{z} = f(\mathbf{n}+1)\ \mathbf{a} \qquad\qquad \mathbf{z}^\dagger = \mathbf{a}^\dagger\ f(\mathbf{n}+1)\ \ , \tag{2.40}
$$

the general form of the measure was found in terms of an inverse Mellin transformation[10], [8]. The result can in principal be expanded in θ_0, and at zeroth order one should get (2.31). This was demonstrated in [8] for the fuzzy sphere. In the next section we give the first order correction to the result.

3 The stereographically projected fuzzy sphere

We first review scalar field theory on the sphere. Because of the requirement that local orthogonal frames form coordinate bases we must work with a nonstandard (i.e., nonconformal) metric.

3.1 Scalar field theory on the commutative sphere

First start with lowest order fuzzy sphere. Set the radius equal to one. In terms of embedding coordinates $x_i,\ i = 1,2,3$, the Poisson brackets are

$$
\{x_i, x_j\} = \epsilon_{ijk} x_k\ , \qquad x_1^2 + x_2^2 + x_3^2 = 1 \tag{3.1}
$$

After stereographically projecting to the complex plane

$$\zeta = \frac{x_1 - ix_2}{1 - x_3} \qquad \bar{\zeta} = \frac{x_1 + ix_2}{1 - x_3} \ , \tag{3.2}$$

the Poisson structure is projected to

$$\{\zeta, \bar{\zeta}\} = -i\theta(|\zeta|^2) \ , \qquad \theta(|\zeta|^2) = \frac{1}{2}(1 + |\zeta|^2)^2 \ , \tag{3.3}$$

which can be used to construct the Kähler two form for S^2. We can then map to a constant Poisson structure

$$\{\alpha, \bar{\alpha}\} = -i \ , \tag{3.4}$$

using

$$\zeta = \rho(|\zeta|^2)\ \alpha \qquad \bar{\zeta} = \rho(|\zeta|^2)\ \bar{\alpha} \tag{3.5}$$

The solution for $\rho(|\zeta|^2)$ is not unique. It is

$$\frac{|\zeta|^2}{\rho(|\zeta|^2)^2} = C - \frac{2}{1 + |\zeta|^2} \tag{3.6}$$

Positivity of $\rho(|\zeta|^2)^2$ means that the integration constant satisfies $C \geq 2$. Furthermore, requiring the mapping to be nonsingular for all $|\zeta| < \infty$ fixes $C = 2$, otherwise ρ vanishes at the origin. We shall 'quantize' about the solution $C = 2$. For this solution

$$\alpha = \frac{\sqrt{2}\ \zeta}{\sqrt{1 + |\zeta|^2}} \qquad \bar{\alpha} = \frac{\sqrt{2}\ \bar{\zeta}}{\sqrt{1 + |\zeta|^2}} \tag{3.7}$$

Now set $y^1 = (\bar{\alpha} + \alpha)/\sqrt{2}$ and $y^2 = i(\bar{\alpha} - \alpha)/\sqrt{2}$ and substitute into (2.32) [replacing θ_S/θ_0 by $\theta(|\zeta|^2)$] to get the lowest order inverse zweibein

$$\begin{aligned} h_1 &= \frac{1}{4}\sqrt{1 + |\zeta|^2}\ \ (2 + |\zeta|^2 - \bar{\zeta}^2) \\ h_2 &= -\frac{i}{4}\sqrt{1 + |\zeta|^2}\ (2 + |\zeta|^2 + \bar{\zeta}^2) \ , \end{aligned} \tag{3.8}$$

and then the lowest order inverse metric

$$\begin{pmatrix} g^{\zeta\zeta} & g^{\zeta\bar{\zeta}} \\ g^{\bar{\zeta}\zeta} & g^{\bar{\zeta}\bar{\zeta}} \end{pmatrix} = \frac{1}{4}(1 + |\zeta|^2) \begin{pmatrix} (2 + |\zeta|^2)\zeta^2 & 2 + 2|\zeta|^2 + |\zeta|^4 \\ 2 + 2|\zeta|^2 + |\zeta|^4 & (2 + |\zeta|^2)\bar{\zeta}^2 \end{pmatrix} \tag{3.9}$$

This does not correspond to the usual conformal metric for the sphere, i.e.

$$\begin{pmatrix} g^{ww} & g^{w\bar{w}} \\ g^{\bar{w}w} & g^{\bar{w}\bar{w}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}(1 + |w|^2)^2 \\ \frac{1}{2}(1 + |w|^2)^2 & 0 \end{pmatrix} \tag{3.10}$$

For that one should map tangent vectors $(\partial/\partial\zeta, \partial/\partial\bar{\zeta})$ to some tangent vectors $(\partial/\partial w, \partial/\partial\bar{w})$

$$\frac{\partial}{\partial\zeta} = t^w_{\ \zeta}\ \frac{\partial}{\partial w} + t^{\bar{w}}_{\ \zeta}\ \frac{\partial}{\partial\bar{w}} \qquad \frac{\partial}{\partial\bar{\zeta}} = t^w_{\ \bar{\zeta}}\ \frac{\partial}{\partial w} + t^{\bar{w}}_{\ \bar{\zeta}}\ \frac{\partial}{\partial\bar{w}} \ , \tag{3.11}$$

where

$$\begin{pmatrix} t^w_\zeta & t^w_{\bar\zeta} \\ t^{\bar w}_\zeta & t^{\bar w}_{\bar\zeta} \end{pmatrix} = \frac{1+|w|^2}{2(1+|\zeta|^2)^{3/2}} \begin{pmatrix} 2+|\zeta|^2 & -\zeta^2 \\ -\bar\zeta^2 & 2+|\zeta|^2 \end{pmatrix} \tag{3.12}$$

However this does not correspond to a coordinate map, i.e. $\partial t^w_\zeta/\partial\bar\zeta \neq \partial t^w_{\bar\zeta}/\partial\zeta$, and the inverse metric (3.10) cannot be connected to the local orthogonal frame via a coordinate map (which is needed for the noncommutative generalization). In terms of the inverse metric (3.9) the action for the free scalar field is

$$\mathcal{S}^0_0 = \frac{i}{8}\int \frac{d\zeta\wedge d\bar\zeta}{1+|\zeta|^2}\left\{ 4|\partial\phi|^2 \ + \ (2+|\zeta|^2)\,(\zeta\partial\phi+\bar\zeta\bar\partial\phi)^2 \right\}, \tag{3.13}$$

where again $\partial = \partial/\partial\zeta,\ \bar\partial = \partial/\partial\bar\zeta$. In what follows we look for the lowest order fuzzy corrections to this action.

3.2 Fuzzy stereographic projection

For the fuzzy sphere one promotes the coordinates x_i to operators $\mathbf{x}_i$'s , satisfying commutation relations:

$$[\mathbf{x}_i,\mathbf{x}_j] = i\beta\ \epsilon_{ijk}\mathbf{x}_k\ , \tag{3.14}$$

as well as $\mathbf{x}_i\mathbf{x}_i = 1\!\!1$. When the parameter β has values $\frac{1}{\sqrt{j(j+1)}}$, $j = \frac{1}{2}, 1, \frac{3}{2}, ...$, $\mathbf{x}_i$ has finite dimensional representations, which are simply given by $\mathbf{x}_i = \beta\mathbf{J}_i$, $\mathbf{J}_i$ being the angular momentum matrices. Denote the $N = 2j+1$ states of an irreducible representation Γ^j as usual by $|j,m>$, $m=-j,-j+1,...,j$, spanning Hilbert space H^j. The commutative limit is $j\to\infty$ corresponding to infinite dimensional representations.

The operator analogue of the stereographic projection to a pair of operators $\mathbf{z}$ and $\mathbf{z}^\dagger$ is defined up to an ordering ambiguity. We fix it as follows:

$$\mathbf{z} = (\mathbf{x}_1 - i\mathbf{x}_2)(1-\mathbf{x}_3)^{-1}\ , \qquad \mathbf{z}^\dagger = (1-\mathbf{x}_3)^{-1}(\mathbf{x}_1 + i\mathbf{x}_2) \tag{3.15}$$

We remark that this transformation is nonsingular for all finite values of j since the eigenvalues of $\mathbf{x}_3$ are less than one. We can now define the operator $\Theta(\mathbf{z},\mathbf{z}^\dagger)$ appearing in (1.4). It is diagonal on H^j

$$\Theta(\mathbf{z},\mathbf{z}^\dagger)|j,m> = \theta^j_m|j,m>\ ,$$
$$\theta^j_m = \frac{j(j+1)-m^2-m}{(\sqrt{j(j+1)}-m-1)^2} - \frac{j(j+1)-m^2+m}{(\sqrt{j(j+1)}-m)^2} \tag{3.16}$$

The two terms in (3.16) are eigenvalues of $\mathbf{z}\mathbf{z}^\dagger$ and $\mathbf{z}^\dagger\mathbf{z}$, respectively. An explicit expression for $\Theta(\mathbf{z},\mathbf{z}^\dagger)$ in terms of just $\mathbf{z}\mathbf{z}^\dagger$ was given in [8]. It is

$$\Theta(\mathbf{z},\mathbf{z}^\dagger) = \beta\chi\left(1+\mathbf{z}\mathbf{z}^\dagger - \frac{1}{2}\chi\,(1+\frac{\beta}{2}\mathbf{z}\mathbf{z}^\dagger)\right), \tag{3.17}$$

where

$$\frac{\beta}{2}\chi = 1+\frac{\beta}{2\xi} - \sqrt{\frac{1}{\xi}+\left(\frac{\beta}{2\xi}\right)^2}\ , \qquad \xi = 1+\beta\mathbf{z}\mathbf{z}^\dagger$$

Expanding this result in $1/j$ gives the classical answer plus the next order correction

$$\Theta(\mathbf{z},\mathbf{z}^\dagger) \to \frac{1}{2j}(1+\mathbf{z}\mathbf{z}^\dagger)^2 - \frac{1}{4j^2}(1+\mathbf{z}\mathbf{z}^\dagger)^3\;, \qquad \text{as}\;\; j \to \infty\;. \tag{3.18}$$

We next define the map from the harmonic oscillator algebra. This is clearly a singular map since H^j is finite dimensional and the Hilbert space $\mathtt{H}$ for the latter is not. For irreducible representation Γ^j, we can restrict the map to act on the finite dimensional subspace of $\mathtt{H}$ spanned by the first $2j+1$ eigenstates $|n>$, $n = 0,1,2,...,2j$, of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$. More precisely, we identify $|j,m>$ in H^j with $|j+m>$ of $\mathtt{H}$, and the map is applied to this subspace. For the map we take the ansatz (2.40). Because the function f depends on j we include a j subscript

$$\mathbf{z} = f_j(\mathbf{n}+1)\;\mathbf{a} \tag{3.19}$$

Using

$$\Theta(\mathbf{z},\mathbf{z}^\dagger) = (\mathbf{n}+1)f_j(\mathbf{n}+1)^2 - \mathbf{n}f_j(\mathbf{n})^2\;, \tag{3.20}$$

it follows that

$$f_j(\mathbf{n}) = \frac{\sqrt{2j-\mathbf{n}+1}}{\sqrt{j(j+1)}+j-\mathbf{n}} \tag{3.21}$$

$f_j(\mathbf{n})$ is zero when acting on $|2j+1>$, and hence $\mathbf{z}^\dagger|2j>=0$. Therefore $\mathbf{z}$ and $\mathbf{z}^\dagger$ have a well defined action on the first $N = 2j+1$ harmonic oscillator states, corresponding to the physical Hilbert space for the fuzzy sphere, and are ill-defined on states with $n > 2j+1$.

On states with $n \le 2j$, $\mathbf{z}$ goes like

$$\mathbf{z} \to \mathbf{a}\;\frac{1}{\sqrt{2j-\mathbf{n}}}\;\biggl(1\;+\;\mathcal{O}(1/j^2)\biggr)\;, \tag{3.22}$$

in the large j limit. The inverse of (3.19) cannot be well defined since $\mathbf{a}^\dagger$ takes vectors out of the physical Hilbert space. As pointed out previously, we only need that the commutators of $\mathbf{a}$ and $\mathbf{a}^\dagger$ with the fields be well defined, and this can be arranged by imposing suitable boundary conditions on the fields. Fields Φ are defined in the enveloping algebra generated by $\mathbf{z}$ and $\mathbf{z}^\dagger$ and hence are nonvanishing on the same N-dimensional subspace of $\mathtt{H}$. In order for the derivatives $\nabla\Phi$ and $\bar{\nabla}\Phi$ to be defined on the same subspace we need $<2j|\Phi = 0$ and $\Phi|2j>=0$, respectively. For the free field action S_0 in (1.3) we then need both of these conditions. S_0 with these boundary conditions is then the free field action on the fuzzy disc.[3] It also serves to approximate the free field action S on the fuzzy sphere in the limit $n << j$ as we show in the next subsection.

Although the inverse of (3.19) is not well defined, we know from (3.7) that the commutative analogue (3.7) in a local coordinate patch containing the origin. With this in mind, we attempt to make an asymptotic expansion of $\mathbf{a}$ and $\mathbf{a}^\dagger$ as a function of $\mathbf{z}$ and $\mathbf{z}^\dagger$. First write $f_j(\mathbf{n})$ in terms of just $\mathbf{z}\mathbf{z}^\dagger$ using

$$\mathbf{n} = j + \sqrt{j(j+1)} - \frac{2}{\beta\chi} \tag{3.23}$$

Upon expanding in $1/j$

$$f_j(\mathbf{n}+1)^2 \to \frac{1}{2j}\,(1+\mathbf{z}\mathbf{z}^\dagger) + \mathcal{O}(1/j^3)\ , \qquad \text{as}\ \ j\to\infty \tag{3.24}$$

Then from (3.19)

$$\mathbf{a} \to \frac{\sqrt{2j}}{\sqrt{1+\mathbf{z}\mathbf{z}^\dagger}}\,\mathbf{z} + \mathcal{O}(1/j^{3/2})\ , \qquad \text{as}\ \ j\to\infty\ , \tag{3.25}$$

which reveals that we have the fuzzyfied about the $C=2$ solution (3.7).

3.3 Star product independent correction

Here we compute $\mathcal{S}_1^0$. We start with the standard action for a field Φ on fuzzy S^2.

$$S = -\frac{\pi}{2j+1}\ \mathrm{Tr}\ [\mathbf{J}_i,\Phi]^2 \tag{3.26}$$

It can be re-expressed on the truncated harmonic oscillator Hilbert space using the Holstein-Primakoff map[12]

$$\mathbf{J}_+ = \sqrt{2j-\mathbf{n}+1}\ \mathbf{a}^\dagger \qquad \mathbf{J}_- = \mathbf{a}\ \sqrt{2j-\mathbf{n}+1} \qquad \mathbf{J}_3 = \mathbf{n}-j \tag{3.27}$$

Just as with the raising operator $\mathbf{z}^\dagger$, $\mathbf{J}_+$ annihilates the highest state, $\mathbf{J}_+|2j>=0$. Substituting into (3.26) gives

$$S = -\frac{\pi}{2j+1}\ \mathrm{Tr}\ \Bigl([\mathbf{a}^\dagger\sqrt{2j-\mathbf{n}},\Phi]\ [\sqrt{2j-\mathbf{n}}\ \mathbf{a},\Phi]\ +\ [\mathbf{n},\Phi]^2\Bigr) \tag{3.28}$$

To recover (1.3), consider the case of Φ vanishing on all states $|n>$ with n greater than some $n_0 << j$. (1.3) appears in the limit of large j. The next order term S_1 in the expansion (2.15) in $1/j$ is

$$\begin{aligned} S_1 &= \frac{\pi}{4j}\ \mathrm{Tr}\ \Bigl([\mathbf{n}\mathbf{a}^\dagger,\Phi][\mathbf{a},\Phi] + [\mathbf{a}^\dagger,\Phi][\mathbf{a}\mathbf{n},\Phi] - 2[\mathbf{n},\Phi]^2\Bigr) \\ &= \frac{\pi}{4j}\ \mathrm{Tr}\ \Bigl(2\nabla\Phi\bar{\nabla}\Phi - 2\mathbf{n}[\nabla\Phi,\bar{\nabla}\Phi] - (\mathbf{a}\nabla\Phi)^2 - (\mathbf{a}^\dagger\bar{\nabla}\Phi)^2\Bigr) \end{aligned} \tag{3.29}$$

This can be expressed as an integral over the symbols α and $\bar{\alpha}$ of $\mathbf{a}$ and $\mathbf{a}^\dagger$. At lowest order in $1/j$ one gets

$$\mathcal{S}_1^0 = \frac{i}{4j}\int d\alpha\wedge d\bar{\alpha}\ \left\{\frac{\partial\phi}{\partial\alpha}\frac{\partial\phi}{\partial\bar{\alpha}} - \frac{\alpha^2}{2}\left(\frac{\partial\phi}{\partial\alpha}\right)^2 - \frac{\bar{\alpha}^2}{2}\left(\frac{\partial\phi}{\partial\bar{\alpha}}\right)^2\right\}\ , \tag{3.30}$$

where Φ is again the symbol of Φ. This is the correction to the scalar field action in the local orthogonal frame $\mathcal{O}$. Since it is the lowest order result it is independent of the star product. Finally, applying the inverse map back the coordinate patch $\mathcal{P}$, one gets the following first order correction to (3.13)

$$\mathcal{S}_1^0 = \frac{i}{16j}\int \frac{d\zeta\wedge d\bar{\zeta}}{(1+|\zeta|^2)^2}\ \left\{4(1+3|\zeta|^2)|\partial\phi|^2 - (2+|\zeta|^2+|\zeta|^4)(\zeta\partial\phi+\bar{\zeta}\bar{\partial}\phi)^2\ \right\} \tag{3.31}$$

3.4 Star product dependent correction

We next compute the star product dependent first order correction $\mathcal{S}_0^1$ to (3.13). We again consider the Voros star product, along with the generalized star product of [8]. However now we need to modify the coherent states by making a finite truncation of the sum over harmonic oscillator states, and as a result, we modify the corresponding star products. The reason for the truncation is because $\mathbf{z}$ and $\mathbf{z}^\dagger$ are only defined on the $2j+1$-dimensional subspace of $\mathtt{H}$. The resulting coherent states will no longer be eigenstates of either $\mathbf{a}$ or $\mathbf{z}$, although they tend to eigenstates in the commutative limit. Our procedure requires a star product written directly on the coordinate patch, and therefore excludes those such as [13] for the sphere which are expressed in terms of embedding coordinates.

3.4.1 Truncated Voros star product

In [11] we examined the star product based on a truncation of the standard coherent states. Here we truncate at $(2j)^{th}$ excited state of the harmonic oscillator, and denote the resulting coherent states by $|\alpha, j>_V$,

$$|\alpha, j>_V = N_{V,j}(|\alpha|^2)^{-\frac{1}{2}} \sum_{n=0}^{2j} \frac{1}{n!}(\alpha \mathbf{a}^\dagger)^n |0> \quad , \tag{3.32}$$

$|0>$ being the harmonic oscillator ground state. The requirement that $|\alpha, j>_V$ are unit vectors fixes $N_{V,j}(|\alpha|^2)$:

$$N_{V,j}(|\alpha|^2) = \sum_{n=0}^{2j} \frac{1}{n!}|\alpha|^{2n} \equiv \mathrm{e}_{2j}(|\alpha|^2) \ . \tag{3.33}$$

The truncated coherent state is almost (up to the harmonic oscillator state $|2j>$) an eigenstate of $\mathbf{a}$

$$\mathbf{a}|\alpha, j>_V = \alpha|\alpha, j>_V \ - \frac{\alpha^{2j+1}}{\sqrt{(2j)!\ \mathrm{e}_{2j}(|\alpha|^2)}}\ |2j> \tag{3.34}$$

As a result

$$_V<\alpha, j|\mathbf{a}|\alpha, j>_V = \alpha\ (1-\mu_j(|\alpha|^2)\ ) \qquad\qquad _V<\alpha, j|\mathbf{a}^\dagger|\alpha, j>_V = \bar{\alpha}\ (1-\mu_j(|\alpha|^2)\ )$$

$$\mu_j(x) = \frac{x^{2j}}{(2j)!\ e_{2j}(x)} \tag{3.35}$$

Below we shall only consider the lowest order corrections to the classical result, which are of order $1/j$. Since $\mu_j(x)$ vanishes much more rapidly than that we are then justified in approximating α and $\bar{\alpha}$ as the covariant symbols of $\mathbf{a}$ and $\mathbf{a}^\dagger$, respectively. Moreover, the corrections to Voros star product will be negligible. The exact expression is obtained from (2.27). The scalar product appearing there can be written as a truncated exponential e_{2j}. For large j (compared to its argument) it rapidly approaches the scalar product of the usual

coherent states. The same is true for the integration measure $d\mu_V(\alpha,\bar\alpha)$. The exact expression was computed in [11]. The result is

$$d\mu_V(\alpha,\bar\alpha) = \frac{i}{2\pi}\Theta_{2j}(|\alpha|^2)\; d\alpha\wedge d\bar\alpha \;, \tag{3.36}$$

with

$$\Theta_N(|\alpha|^2) = \frac{\Gamma[N+1,|\alpha|^2]}{\Gamma[N+1]} = e^{-|\alpha|^2} e_N(|\alpha|^2)\;, \tag{3.37}$$

where $\Gamma[N,x]$ denotes the incomplete gamma function. For $j >> 1$ the difference of Θ_{2j} and 1 is exponentially small.[‡]

Next we use (3.22) to compute the covariant symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$ for large j. The result is

$$\zeta(\alpha,\bar\alpha)\to\frac{\alpha}{\sqrt{2j-|\alpha|^2}}\left(1\;+\;\frac{j-\frac18|\alpha|^2}{(2j-|\alpha|^2)^2}\;+\;\mathcal{O}(1/j^2)\right)\;, \tag{3.38}$$

with the covariant symbol of $\mathbf{z}^\dagger$ being the complex conjugate. Since (3.22) is only valid for $n\le 2j$, (3.38) will be only valid for $|\alpha|^2\le 2j$. Upon inverting this expression we get the first order correction to the commutative result (3.7)

$$\alpha(\zeta,\bar\zeta) = \frac{\sqrt{2j}\;\zeta}{\sqrt{1+|\zeta|^2}}\left(1-\frac{1}{16j}(4+3|\zeta|^2)+\mathcal{O}(1/j^2)\right) \tag{3.39}$$

The correction is small provided $|\zeta|^2 << j$. It remains to compute the inverse zweibein matrix (2.22). We get

$$\begin{aligned} h^\zeta{}_\alpha &= h^{\bar\zeta}{}_{\bar\alpha} = \sqrt{\frac{1+|\zeta|^2}{2j}}\left\{1+\frac{|\zeta|^2}{2}+\frac{\eta}{16j}\;+\mathcal{O}(1/j^2)\right\}\\ h^\zeta{}_{\bar\alpha} &= \overline{h^{\bar\zeta}{}_\alpha} = \frac{\zeta^2}{2}\sqrt{\frac{1+|\zeta|^2}{2j}}\left\{1+\frac{\xi}{16j}+\mathcal{O}(1/j^2)\right\}\;, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} \eta &= 4+8|\zeta|^2+\frac{15}{2}|\zeta|^4+3|\zeta|^6\\ \xi &= 10+15|\zeta|^2+6|\zeta|^4 \end{aligned} \tag{3.41}$$

The result can then be substituted into (2.23) and the measure to give the correction $\mathcal{S}^1_0$. Keeping the first order corrections to the $n=0$ contribution to the sum in (2.23) we get

$$|\;(h^\zeta{}_\alpha\partial+h^{\bar\zeta}{}_\alpha\bar\partial)\;\phi\;|^2\quad\to\quad\frac{(1+|\zeta|^2)}{2j}\left\{|\partial\phi|^2+\frac14(2+|\zeta|^2)(\zeta\partial\phi+\bar\zeta\bar\partial\phi)^2\right. \tag{3.42}$$

[‡]Another interesting limit of Θ_N was found in [11],[3]. It is the limit of the disc. For this one re-scales the coordinates α and $\bar\alpha$ by a factor of $1/\sqrt{\theta_0}$ and sets $N\to\infty$ and $\theta_0\to 1/N$. Then Θ_N goes to the characteristic function on a unit disc:

$$\Theta_N(|\alpha|^2/\theta_0)\to\begin{cases}1\,, & |\alpha|<1\\ 0\,, & |\alpha|>1\end{cases}$$

$$+\frac{1}{16j}\biggl((2\eta-|\zeta|^2\xi)|\partial\phi\,|^2+\frac{1}{4}(2\eta+2\xi+|\zeta|^2\xi)(\zeta\partial\phi+\bar\zeta\bar\partial\phi)^2\biggr)\ \Bigr\}\ ,$$

while the lowest order correction to the $n=1$ term is

$$\begin{aligned}|\ (h^{\zeta}_{\ \alpha}\partial+h^{\bar\zeta}_{\ \alpha}\bar\partial)^2\ \phi\ |^2 \quad\to\quad & \frac{(1+|\zeta|^2)^2}{64j^2}\Bigl|\,3\zeta^3\ \partial\phi\ +\ (4+3|\zeta|^2)\zeta\ \bar\partial\phi \\ & +\zeta^4\ \partial^2\phi\ +\ 2\zeta^2(2+|\zeta|^2)\ \partial\bar\partial\phi\ +\ (2+|\zeta|^2)^2\ \bar\partial^2\phi\ \Bigr|^2\end{aligned} \tag{3.43}$$

Up to first order the determinant becomes

$$\det h_V=\frac{1}{2j}(1+|\zeta|^2)^2\ \left[1+\frac{1}{16j}\ \frac{(2+|\zeta|^2)\eta-\frac{1}{2}|\zeta|^4\xi}{1+|\zeta|^2}\right] \tag{3.44}$$

Combining these results gives

$$\begin{aligned}\mathcal{S}^1_0\ \ =\ \ & \frac{i}{32j}\int d\zeta\wedge d\bar\zeta\ \ \Bigl\{\frac{|\zeta|^2(\eta-\xi)-\frac{1}{2}|\zeta|^4\xi}{(1+|\zeta|^2)^2}\ |\partial\phi|^2 \\ & +\ \frac{2(\xi-\eta)+(3\xi-2\eta)\ |\zeta|^2+(2\xi-\eta)|\zeta|^4-\frac{1}{2}\xi|\zeta|^6}{4(1+|\zeta|^2)^2}\ (\zeta\partial\phi+\bar\zeta\bar\partial\phi)^2 \\ & +\ \frac{1}{2}\,\Bigl|\ 3\zeta^3\partial\phi+(4+3|\zeta|^2)\zeta\bar\partial\phi+\zeta^4\partial^2\phi+2\zeta^2(2+|\zeta|^2)\partial\bar\partial\phi+(2+|\zeta|^2)^2\bar\partial^2\phi\ \Bigr|^2\ \Bigr\}\end{aligned} \tag{3.45}$$

3.4.2 Truncated generalized star product

We denoted the truncated generalized coherent states by $|\zeta,j>$ in [8]. They were written as a finite series of eigenstates of $\mathbf{n}$, and as a result are no longer eigenstates of $\mathbf{z}$. Instead,

$$\mathbf{z}|\zeta,j>=\zeta|\zeta,j>-\frac{N_j(|\zeta|^2)^{-\frac{1}{2}}\ \zeta^{2j+1}}{\sqrt{(2j)!\ [f_j(2j)]!}}\ |2j>\ , \tag{3.46}$$

where $[f(n)]!=f(n)f(n-1)...f(0)$ and $N_j(x)$ is a normalization function. From the demand that $|\zeta,j>$ has unit norm,

$$N_j(x)=\sum_{n=0}^{2j}\frac{x^n}{n!\ ([f_j(n)]!)^2}\ , \tag{3.47}$$

which can be expressed in terms of a hypergeometric function

$$N_j(x)=\ \frac{\Gamma(\gamma+2j+1)\ ^2}{(2j+1)!\ (2j)!\ \Gamma(\gamma)\ ^2}\ \ {}_3F_2(1,1,-2j;\gamma,\gamma;-x^{-1})\ x^{2j}\ , \tag{3.48}$$

where $\gamma=\sqrt{j(j+1)}-j$. So now ζ and $\bar\zeta$ are not covariant symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$, respectively. Rather,

$$<\zeta,j|\mathbf{z}|\zeta,j>=\zeta\ (1-\nu_j(|\zeta|^2)\ )\qquad\qquad <\zeta,j|\mathbf{z}^\dagger|\zeta,j>=\bar\zeta\ (1-\nu_j(|\zeta|^2)\ )$$

$$\nu_j(x)=\frac{x^{2j}}{(2j)!\ ([f_j(2j)]!)^2N_j(x)} \tag{3.49}$$

Next we expand in $1/j$. The asymptotic behavior of $N_j(x)$ is

$$N_j(x) \sim (1+x)^{2j} \left(\frac{2j}{1+x}\right)^{2(1-\gamma)} \exp\left(\frac{1+x}{8j}\right) , \tag{3.50}$$

which is valid for $|\zeta|^2 = x \ll j$. Using $(2j)!\ ([f_j(2j)]!)^2 \sim 2\pi j$, $\nu_j(x)$ goes like

$$\nu_j(x) \sim \frac{1}{4\pi j^2} \frac{x^{2j}\ e^{-\frac{1+x}{8j}}}{(1+x)^{2j-1}} \ , \qquad \text{as } j \to \infty \tag{3.51}$$

Below we shall only consider the lowest order corrections to the classical result, which are of order $1/j$. Therefore the correction $\nu_j(x)$ can be ignored, and we are justified in approximating ζ and $\bar{\zeta}$ as the covariant symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$, respectively. Using these coherent states we can directly compute the symbols of $\Theta(\mathbf{z},\mathbf{z}^\dagger)$ and $\mathbf{a}$, which was not the case using the Voros product. So from (3.18) the symbol $\theta_S(\zeta,\bar{\zeta})$ for $\Theta(\mathbf{z},\mathbf{z}^\dagger)$ can be approximated up to the first order corrections by

$$\theta_S(\zeta,\bar{\zeta}) \quad \rightarrow \quad \frac{1}{2j}(1+\zeta\star\bar{\zeta})^{\star 2} - \frac{1}{4j^2}(1+\zeta\star\bar{\zeta})^{\star 3} \ , \qquad \text{as } j \to \infty \tag{3.52}$$

If we apply the expansion (2.29) for the star product this leads to

$$\theta_S(\zeta,\bar{\zeta}) \quad = \quad \frac{1}{2j}(1+|\zeta|^2)^2 \left(1 + \frac{1}{2j}(1+2|\zeta|^2) + \mathcal{O}(1/j^2)\right) \tag{3.53}$$

The lowest order term corresponds to $\frac{1}{j}\theta(|\zeta|^2)$ in (3.3). If we again use α to denote the symbol of $\mathbf{a}$, then from (3.25) we get

$$\alpha = \sqrt{2j}\ (1+\zeta\star\bar{\zeta})^{\star(-\frac{1}{2})} \star \zeta + \mathcal{O}(1/j^{3/2}) \tag{3.54}$$

Then we can expand the definition $\mathcal{A}^{\star(-\frac{1}{2})}\star\mathcal{A}^{\star(-\frac{1}{2})}\star\mathcal{A} = 1$ to compute the first order correction to (3.7)

$$\alpha = \frac{\sqrt{2j}\ \zeta}{\sqrt{1+|\zeta|^2}} \left(1 - \frac{1}{16j}(4+|\zeta|^2) + \mathcal{O}(1/j^2)\right) , \tag{3.55}$$

which differs slightly from the analogous expression (3.39) obtained using the Voros star product. Once again the region of validity is $|\zeta|^2 << j$.

In the above we have used the star product expansion (2.29) up to order $1/j$. However to obtain the leading correction to the commutative action we must expand $y^a \star \phi$ and $\phi \star y^a$ in (2.30) up to $1/j^2$ since the difference, or star commutator, appears there. Because ζ and $\bar{\zeta}$ are symbols of $\mathbf{z}$ and $\mathbf{z}^\dagger$ only up to order $1/j^2$, there will be corrections to the star product expansion (2.29) up to this order. For example, while the term

$$\mathcal{A}(\zeta,\bar{\zeta}) \ \int d\mu(\eta,\bar{\eta})\ \frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta) \quad <\zeta,j|\eta,j><\eta,j|\zeta,j> \quad \mathcal{B}(\zeta,\bar{\zeta}) \tag{3.56}$$

vanishes when ζ is the symbol of $\mathbf{z}$, it now produces $1/j^2$ corrections. Using (3.46) this becomes

$$\frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar{\zeta})\ \mathcal{B}(\zeta,\bar{\zeta}) \ \int d\mu(\eta,\bar{\eta}) \Big\{ <\zeta,j|\mathbf{z}|\eta,j><\eta,j|\zeta,j> - <\zeta,j|\eta,j><\eta,j|\mathbf{z}|\zeta,j>$$

$$+\ \frac{1}{\sqrt{(2j)!}\ [f_j(2j)]!}\biggl(\ \frac{\eta^{2j+1}}{N_j(|\eta|^2)^{\frac12}} < \zeta, j|2j >< \eta, j|\zeta, j >$$

$$-\ \frac{\zeta^{2j+1}}{N_j(|\zeta|^2)^{\frac12}} < \zeta, j|\eta, 2j >< \eta, j|2j >\biggr)\ \biggr\} \tag{3.57}$$

The first line in the braces vanishes by the partition of unity. Using

$$< \eta, j|\zeta, j >= N_j(|\eta|^2)^{-\frac12}\ N_j(|\zeta|^2)^{-\frac12} N_j(\bar\eta\zeta)\ , \tag{3.58}$$

we get

$$\frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar\zeta)\ \ \mathcal{B}(\zeta,\bar\zeta)\ \biggl[\ \int d\mu(\eta,\bar\eta)\ \frac{\bar\zeta^{2j}\eta^{2j+1}\ N_j(\bar\eta\zeta)}{(2j)!\ ([f_j(2j)]!)^2\ N_j(|\zeta|^2)\ N_j(|\eta|^2)}\ \ -\ \ \zeta\nu_j(|\zeta|^2)\ \biggr]\ , \tag{3.59}$$

which goes like $1/j^2$. Analogous $1/j^2$ corrections come from

$$\mathcal{A}(\zeta,\bar\zeta)\ \ \int d\mu(\eta,\bar\eta)\ \ < \zeta, j|\eta, j >< \eta, j|\zeta, j >\ \ (\bar\eta-\bar\zeta)\frac{\overrightarrow{\partial}}{\partial\bar\zeta}\ \ \mathcal{B}(\zeta,\bar\zeta) \tag{3.60}$$

So upon expanding $y^a\star\phi$ and $\phi\star y^a$ in the Lagrangian (2.30) up to $1/j^2$ we get terms involving single derivatives of ϕ, as well as terms proportional to ϕ^2. Such terms did not appear using the truncated Voros star product. There will also be corrections to the coefficients of the quadratic terms (2.34).

Next we write down corrections to the classical measure. We set

$$\frac{2\pi}{j}\ d\mu(\zeta,\bar\zeta) = H_j(|\zeta|^2)\ d\zeta\wedge d\bar\zeta\ , \tag{3.61}$$

corresponding to the left hand side of (2.31). In [8] we found an exact expression for $H_j(x)$ in terms of hypergeometric function ${}_2F_1$:

$$H_j(x) = \frac{i}{j}\ \ N_j(x)\ \ {}_2F_1(\gamma+2j+1,\gamma+2j+1;2j+2;-x), \tag{3.62}$$

where again $\gamma=\sqrt{j(j+1)}-j$ and $N_j(x)$ was given in (3.47) and (3.48). The asymptotic expansion of ${}_2F_1$ for large parameters [14] is

$${}_2F_1(a_1+2j,a_2+2j;b+2j;-x)\sim(1+x)^{b-a_1-a_2-2j}\biggl(1\ -\ \frac{(b-a_1)(b-a_2)x}{2j}\ +\ \mathcal{O}(1/j^2)\biggr)\ , \tag{3.63}$$

while the asymptotic form of $N_j(x)$ was given in (3.50). Both of these expressions are only valid for $x\ll j$. Then

$$H_j(x)\sim\frac{2i}{(1+|\zeta|^2)^2}\ \biggl(1+\frac{1+2\ln 2j}{8j}+\mathcal{O}(1/j^2)\biggr) \tag{3.64}$$

In what remains we compute the coefficients of the cubic and quartic terms which are unaffected by the above considerations. Call y^a the symbol for Y^a, and set $\theta_0=1/j$. Then from (2.13)

$$\alpha=\sqrt{\frac{j}{2}}\ (y^1+iy^2) \tag{3.65}$$

We can then use (2.32) to compute $1/j$ corrections to the inverse zweibein (3.8)

$$\begin{aligned} h_1 &= \frac{1}{4}\sqrt{1+|\zeta|^2}\ \Biggl\{2+|\zeta|^2+\frac{1}{16j}(8+32|\zeta|^2+13|\zeta|^4) \\ &\qquad - \bar\zeta^2\biggl(1+\frac{1}{16j}(6+17|\zeta|^2)\biggr)\Biggr\}+\mathcal{O}(1/j^2) \\ h_2 &= -\frac{i}{4}\sqrt{1+|\zeta|^2}\ \Biggl\{2+|\zeta|^2+\frac{1}{16j}(8+32|\zeta|^2+13|\zeta|^4) \\ &\qquad + \bar\zeta^2\biggl(1+\frac{1}{16j}(6+17|\zeta|^2)\biggr)\Biggr\}+\mathcal{O}(1/j^2) \end{aligned} \tag{3.66}$$

For the coefficients (2.36) and (2.38) of the cubic and quartic terms, respectively, of scalar field theory we get

$$\begin{aligned} G^{\zeta,\zeta\zeta} &= \frac{1}{8j}\,\zeta^3\,(1+|\zeta|^2)^2\,(3+2|\zeta|^2) \\ G^{\bar\zeta,\zeta\zeta} &= \frac{1}{8j}\,\zeta\,(1+|\zeta|^2)^2\,(4+6|\zeta|^2+3|\zeta|^4) \\ G^{\zeta,\zeta\bar\zeta} &= \frac{1}{8j}\,\zeta\,(1+|\zeta|^2)^3\,(4+3|\zeta|^2) \\ G^{\zeta\zeta,\bar\zeta\bar\zeta} &= \frac{1}{8j}\,(1+|\zeta|^2)^3\,(2+2|\zeta|^2+|\zeta|^4) \\ G^{\bar\zeta\bar\zeta,\zeta\bar\zeta} &= \frac{1}{8j}\,\bar\zeta^2\,(1+|\zeta|^2)^3\,(2+|\zeta|^2)\,, \end{aligned} \tag{3.67}$$

along with their complex conjugates. These results looks quite different from the leading cubic and quartic terms obtained in (3.43) from the Voros star product.

4 Concluding Remarks

Here we remark on possible generalizations of this work.

Although messy it is straightforward to go beyond the first order corrections. Three different contributions occur at second order in θ_0. They are: $\mathcal{S}^0_2$, $\mathcal{S}^1_1$, $\mathcal{S}^2_0$. The first term is star product independent, while the last requires expanding the star product to second order. Also one can try other star products on the plane, such as those developed in [15], [16]. As noted previously, our procedure requires a star product written directly on the coordinate patch.

Another obvious generalization is to go to more than two dimensions. Since we need a nonsingular Poisson structure we should restrict to an even number of dimensions $\mathtt{d}$. For $\mathtt{d} > 2$ we must distinguish between the set of all orthogonal frames $\{\mathcal{O}\}$ and the set of orthogonal frames, which we denote by $\{\tilde{\mathcal{O}}\}$, with a constant Poisson structure, i.e. the Poisson tensor has constant components $\tilde\theta^{ab}$. We cannot identify $\{\mathcal{O}\}$ with $\{\tilde{\mathcal{O}}\}$ because the constant Poisson tensor is not preserved under general local orthogonal transformations (2.3). Instead $\{\tilde{\mathcal{O}}\} \subset$

$\{\mathcal{O}\}$, and the two sets of frames are related by local othogonal transformations. Now denote by $\tilde{e}^c{}_\mu(x)$ those veilbeins which transform from the coordinate patch $\mathcal{P}$ to $\tilde{\mathcal{O}}$. For dimension greater than two, the set of such veilbeins $\{\tilde{e}^a{}_\mu(x)\}$ is a subset of the set of all veilbeins $\{e^a{}_\mu(x)\}$. If the basis vectors of $\tilde{\mathcal{O}}$ are $\frac{\partial}{\partial \tilde{y}^a}$, then

$$\frac{\partial}{\partial x^\mu} = \tilde{e}^a{}_\mu(x)\, \frac{\partial}{\partial \tilde{y}^a} \tag{4.1}$$

Since we again need coordinate maps to flat noncommutative manifolds, the frames $\{\tilde{\mathcal{O}}\}$ need to be coordinate bases. Calling its coordinates $\tilde{y}^a$,

$$\{\tilde{y}^a, \tilde{y}^b\} = \tilde{\theta}^{ab} = \text{constants} \tag{4.2}$$

As $\tilde{\theta}^{ab}$ is nonsingular, using (4.1), we get

$$\frac{\partial}{\partial x^\mu} = \tilde{e}^a{}_\mu(x)\; [\tilde{\theta}^{-1}]_{ab}\; \{\tilde{y}^b\;,\;\;\} \tag{4.3}$$

Then the action of a massless scalar field ϕ in a local region σ of $\mathbb{R}^{\mathbf{d}}$ is

$$\mathcal{S}_0 = \frac{1}{2} \int_\sigma d^{\mathbf{d}}y\; [\tilde{\theta}^{-1}]_{ab}\;\; [\tilde{\theta}^{-1}]_{ac}\; \{\tilde{y}^b\;, \phi\}\; \{\tilde{y}^c\;, \phi\}\;, \tag{4.4}$$

which is easily generalized to the noncommutative case. It remains to find the maps from $\mathbf{x}^\mu$ to $\tilde{\mathbf{y}}^a$, the noncommutative analogues of x^μ to $\tilde{y}^a$, respectively, and re-express the noncommutative action in terms of the symbols of $\mathbf{x}^\mu$.

It would also be of interest to go beyond scalar field theories. In addition to including a mass and interaction term in the scalar field theory is the possibility of changing the target space. For example, one can investigate fuzzy corrections to the nonlinear σ-model and its soliton solutions. A more challenging generalization involves the inclusion of spin. For this we need the analogue of a spin connection $[\omega_\mu]^a{}_b(x)$. The covariant derivative of components u^a of a vector V in a local orthogonal frame $\mathcal{O}$ is given by

$$D_\mu u^a = \frac{\partial}{\partial x^\mu} u^a + [\omega_\mu]^a{}_b u^b \tag{4.5}$$

Upon transforming to another local orthogonal frame $\mathcal{O}'$

$$D_\mu u^a \rightarrow [D_\mu u]'^a = \lambda^a{}_b(x) D_\mu u^b\;, \tag{4.6}$$

where $[\omega_\mu]^a{}_b(x)$ transforms as

$$\omega_\mu \rightarrow \omega'_\mu = \lambda \omega_\mu \lambda^{-1} - \frac{\partial}{\partial x^\mu} \lambda\; \lambda^{-1} \tag{4.7}$$

So here in addition to the noncommutative maps, which play the role of noncommutative veilbeins, one needs the noncommutative analogue of the spin connections. The latter requirement should be similar to having a Dirac operator.

Up to now we have examined a single coordinate patch. It is natural to ask whether the full noncommutative manifold can be described in terms of coordinate patches. For this it is necessary to define the analogue of transition functions on overlapping patches. It is possible that the procedure can be used to define new noncommutative manifolds.

A final possibility we mention is to make the analogue of the veilbeins and spin connection dynamical and thus move in the direction of a noncommutative general relativity. A step in this direction is to consider dynamical maps from operators $\mathbf{x}^i$ satisfying an arbitrary noncommutative algebra to flat noncommutative manifolds spanned by operators $\mathbf{y}^i$. This may be facilitated as in [17] with the introduction of dynamical fields $A_i(\mathbf{x})$, writing the map as $\mathbf{x}^i = \mathbf{y}^i + \theta^{ij} A_j(\mathbf{x})$. The goal would then be to integrate over all maps, i.e. fields $A_i(\mathbf{x})$, and spin connections $[\omega_\mu]^a{}_b(x)$ or Dirac operators.

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